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DEPENDENCE OF KAZHDAN CONSTANTS ON GENERATING SUBSETS

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ABSTRACT

In this paper we answer a question of A. Lubotzky by giving examples of groups having property (T) without uniform Kazhdan constants. We show that many lattices in Lie groups do not admit a Kazhdan constant which is independent of the generating subset.

In 1967 Kazhdan defined property (T). Let Γ be a discrete group generated by a finite set S. The group Γ has Kazhdan property (T) (or, briefly speaking, Γ is a Kazhdan group) if there exists a positive constant $\varepsilon(S)$ such that for every unitary representation $(\pi, \mathcal{H})$ of Γ with no invariant vectors and for any $u \in \mathcal{H}$ there exists $s \in S$ such that $\|\pi(s)u - u\| \geq \varepsilon(S)\|u\|$. Such a constant is called a Kazhdan constant with respect to the set S. The fact that a given group has property (T) does not depend on the set of generators, i.e., if for some finite set of generators S there exists a positive constant $\varepsilon(S)$, then for any finite set

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of generators S' there exists a positive constant $\varepsilon(S')$. There are few explicit computations or estimates of Kazhdan constants (see [1], [2], [3], [4], [13], [15], [16], [18]).

A. Lubotzky asked ([8] page 126) whether for a group with property (T) there exists a uniform Kazhdan constant $\varepsilon > 0$ which is independent of S. The question about uniform Kazhdan constants is related to other problems. For example, it is closely related to questions about the uniform exponential growth (see [6]). It is easy to see that if an infinite group Γ has uniform Kazhdan constant, then it has uniform exponential growth (see [15]). Other related questions are those concerning the independence speculation: the independence of being a family of expanders from the generating set (see [9], see also [8]). A. Lubotzky and A. Wigderson recently gave a counterexample to the independence speculation (see [10]).

In this paper we give first examples of Kazhdan groups without uniform Kazhdan constants.

For a unitary representation $(\pi, \mathcal{H})$ we define the uniform Kazhdan constant $K(\pi, \Gamma)$,

$$K(\pi,\Gamma) = \inf_{S} \inf_{0 \neq u \in \mathcal{H}} \max_{s \in S} \frac{\|\pi(s)u - u\|}{\|u\|},$$

where the infimum is taken over all finite generating sets S. We say that Γ has **uniform property (T)** if $\inf_{\pi} K(\pi, \Gamma) > 0$ where the infimum is taken over all unitary representations with no invariant vectors. We define also

$$K(\pi,\Gamma,k) = \inf_{|S| \leq k} \inf_{0 \neq u \in \mathcal{H}} \max_{s \in S} \frac{\|\pi(s)u - u\|}{\|u\|}$$

(here S runs over the generating sets of size $\leq k$), and we say that Γ has uniform property (T) with respect to generating sets of bounded size, if $K(\pi, \Gamma, k) > 0$ for any $k \geq d(\Gamma)$ (where $d(\Gamma)$ denotes the minimal size of a generating set of Γ) and, of course, for any π.

The purpose of this note is to prove the following

THEOREM: *Let Γ be a Kazhdan group densely embedded (or more generally, which has a dense homomorphic image) in a connected topological group G. Assume that there exists a continuous unitary representation $(\pi_G, \mathcal{H})$ of G without invariant vectors. Then Γ does not have uniform property (T).*

If, moreover, G is a connected Lie group, then Γ does not have uniform property (T) even with respect to generating sets of bounded size.

The assumption on the existence of a continuous unitary representation without invariant vectors is automatically satisfied if G is locally compact. One can take the action by left multiplication on $L^2(G)$, if G is not compact, and on $L_0^2(G)$ (the orthogonal complement to the constant functions), if G is compact.

An example of G and Γ that satisfy the assumption of the theorem is $G = \mathrm{SO}_n(\mathbb{R})$ and $\Gamma = \mathrm{SO}_n(\mathbb{Z}[1/5])$ for $n \geq 5$ (see [11]), or $\Gamma = \mathrm{SO}(q) \cap \mathrm{SL}_n(\mathbb{Z}[\sqrt{2}])$ where q is the quadratic form $q(x) = x_1^2 + \cdots + x_{n-2}^2 - \sqrt{2}(x_{n-1}^2 + x_n^2)$ (see [17]). In both cases Γ is a Kazhdan group, being isomorphic (in a natural way) to a lattice in a Kazhdan Lie group. Margulis and Sullivan used these groups to answer the Banach–Ruziewicz problem about the uniqueness of finitely additive rotation invariant measures defined on all Lebesgue measurable subsets of the n-sphere. As in our situation, they needed dense subgroups with property (T) in Lie groups (in their case just $\mathrm{SO}_n(\mathbb{R})$).

More generally, there are two main families of examples of Kazhdan groups densely mapped into connected simple Lie groups (see [18]):

1. Assume that L is a connected Kazhdan semi-simple Lie group and $\Gamma \leq L$ is a cocompact arithmetic lattice. Γ is commensurable to the projection to L of the group of integral points $\mathbf{H}(\mathbb{Z})$, where $\mathbf{H}$ is a $\mathbb{Q}$-algebraic group with $\mathbf{H}(\mathbb{R}) \cong L \times K$. Assume that the compact group K is non-trivial. The projection of Γ to K is dense and thus we can take G to be any simple factor of K.
2. Any S-arithmetic lattice in a product (of Kazhdan simple groups) for which at least one of the places is archimedean and at least one of the places is non-archimedean and corresponds to an anisotropic group, is densely embedded in a connected Lie group (e.g., each of those that lie in the archimedean places). In some examples this connected group cannot be compact: e.g., $\mathrm{SL}_3(\mathbb{Z}[\frac{1}{p}])$ is naturally densely embedded in $\mathrm{SL}_3(\mathbb{R})$, but every homorphism of $\mathrm{SL}_3(\mathbb{Z}[\frac{1}{p}])$ into a compact connected group has finite image.

Proof of the Theorem: Throughout the proof we will assume, for simplicity, that Γ is actually embedded in G. The proof remains valid for the more general case (where $f\colon \Gamma \to G$ is not necessarily an embedding) if one works with pre-images in Γ of subsets of G instead of intersecting such subsets with Γ.

Let Ω be a neighborhood of the identity in G. As G is connected Ω generates G.

LEMMA 1: *The set $\Gamma \cap \Omega$ contains a finite set S_Ω which generates Γ.*

Proof: First of all $\Gamma \cap \Omega$ generates Γ. Indeed, Γ is dense in G and therefore $\Gamma \cap \Omega$ is dense in Ω. As Ω generates G, $\Gamma \cap \Omega$ generates a dense subgroup in G. This means that for any $\gamma \in \Gamma$ there exists γ' in the group generated by $\Gamma \cap \Omega$

which is in the $\gamma\Omega$ neighborhood of γ, i.e., $(\gamma)^{-1}\gamma' \in \Omega$. But this implies that γ itself is in the group generated by $\Gamma \cap \Omega$.

The group Γ is finitely generated by a finite set S. As $\Gamma \cap \Omega$ generates Γ every element in S can be expressed as a product of a finite number of elements in $\Gamma \cap \Omega$ which provides a finite generating set $S_\Omega \subset \Gamma \cap \Omega$. ∎

Denote by π_Γ the restriction of π_G to Γ. The representation $(\pi_\Gamma, \mathcal{H})$ has no non-zero invariant vectors, because Γ is dense in G and $(\pi_G, \mathcal{H})$ is continuous. Another consequence of the continuity of $(\pi_G, \mathcal{H})$ is that, for any non-zero vector $u \in \mathcal{H}$,

$$\lim_{\Omega \to id} \sup_{g \in \Omega \cap \Gamma} \frac{\|\pi_G(g)u - u\|}{\|u\|} = 0.$$

In particular, by Lemma 1

$$K(\pi_\Gamma, \Gamma) \leq \lim_{\Omega \to id} \max_{s \in S_\Omega} \frac{\|\pi_\Gamma(s)u - u\|}{\|u\|} = 0.$$

This proves the first assertion.

Assume now that G is a connected Lie group. Since Γ is dense and Kazhdan G is non-solvable and therefore, by dividing out the radical, and then mapping to a simple factor, we may assume that G is simple. Let

$$n = \dim(G), \quad d = d(\Gamma) = \text{ the minimal size of a generating set of } \Gamma.$$

We will show that $K(\pi_\Gamma, \Gamma, d+n) = 0$, i.e., that Γ does not have uniform property T with respect to generating sets of size $d + n$. In light of the above argument, it is enough to find generating sets of size $d + n$ in an arbitrarily small identity neighborhood.

Let $\mathfrak{g}$ be the Lie algebra of G. For $X \in \mathfrak{g}$ we denote by $B_r^{\mathfrak{g}}(X)$ the ball of radius r around X in $\mathfrak{g}$.

Assume that ε is sufficiently small so that $B_\varepsilon^{\mathfrak{g}}(0)$ is a Zassenhaus neighborhood in the sense of [14] (see definition 8.22), and that exp: $B_\varepsilon^{\mathfrak{g}}(0) \to B_\varepsilon^G(1)$ is a diffeomorphism. Denote its inverse by

$$\log\colon B_\varepsilon^G(1) \to B_\varepsilon^{\mathfrak{g}}(0).$$

Let $\{X_i\}_{i=1}^n \subset B_\varepsilon^{\mathfrak{g}}(0)$ be a basis for the vector space $\mathfrak{g}$, and let δ be sufficiently small so that

1. $B_\delta^{\mathfrak{g}}(X_i) \subset B_\varepsilon^{\mathfrak{g}}(0)$ for any $1 \leq i \leq n$, and
2. for any selection $Y_i \in B_\delta^{\mathfrak{g}}(X_i)$ we have $\mathfrak{g} = \text{span}\{Y_i\}_{i=1}^n$.

Let $B_i = \exp B_\delta^{\mathfrak{g}}(X_i)$. Take $y_i \in B_i$ and put $Y_i = \log y_i \in B_\delta^{\mathfrak{g}}(X_i)$.

LEMMA 2: *The group $A = \langle y_i : 1 \le i \le n \rangle$ is dense in G.*

Proof: Let N be the identity component of the closure of A, i.e.,

$$N = \overline{A}^0.$$

N is closed, and thus, by a theorem of Cartan, N is a Lie subgroup. Let $\mathfrak{n} \subset \mathfrak{g}$ be its Lie algebra.

First notice that $\dim \mathfrak{n} > 0$. Otherwise A would be discrete, and therefore, by the Kazhdan–Margulis theorem (see [14] theorem 8.16), the y_i's would be inside a connected nilpotent Lie subgroup. But then the Y_i's would generate a nilpotent Lie algebra. This is a contradiction to the fact that $\{Y_i\}$ spans the simple Lie algebra $\mathfrak{g}$.

We claim further that $\mathfrak{n}$ is an ideal in $\mathfrak{g}$. Since $\mathfrak{g} = \operatorname{span}\{Y_i\}$ it is enough to show that $\operatorname{ad}(Y_i)(\mathfrak{n}) = \mathfrak{n}$ for any $1 \le i \le n$, but

$$\operatorname{ad}(Y_i)(\mathfrak{n}) = \operatorname{ad}(\log y_i)(\mathfrak{n}) = (\log \operatorname{Ad}(y_i))(\mathfrak{n}),$$

and since $y_i \in A$, it normalizes $N = \overline{A}^0$, i.e., $\operatorname{Ad}(y_i)(\mathfrak{n}) = \mathfrak{n}$, which implies also that $\log \operatorname{Ad}(y_i)(\mathfrak{n}) = \mathfrak{n}$.

It follows that N is a normal connected subgroup of positive dimension. Since G is simple, $N = G$. Thus A is dense. ∎

Now, since Γ is dense in G we can take the y_i's to be elements in Γ. If $S = \{\gamma_1, \gamma_2, \ldots, \gamma_d\}$ is any generating set for Γ then, by the same argument as in the proof of Lemma 1, we can find $\gamma'_1, \ldots, \gamma'_d \in B^G_\varepsilon(1)$ such that the set $\{y_1, \ldots, y_n, \gamma'_1, \ldots, \gamma'_d\}$ would generate Γ (with products of $y_i^{\pm}$'s we can get as close as needed to γ_j, and with one more element $\gamma'_j \in \Gamma \cap B^G_\varepsilon(1)$ we can actually get γ_j). ∎

Remark: Since the simple Lie algebra $\mathfrak{g}$ is generated by two elements, we can find, in a similar way, a generating set of size $|S| = d(\Gamma) + 2$ with arbitrarily small Kazhdan constant.

There are Kazhdan groups for which our method does not apply, and the question whether they have uniform property (T) remains open. A particularly interesting example is the group $\mathrm{SL}_3(\mathbb{Z})$; specifically, is there a positive uniform Kazhdan constant for all generating sets (or for those of size ≤ 100) of $\mathrm{SL}_3(\mathbb{Z})$? A. Eskin, S. Mozes and H. Oh have recently shown that $\mathrm{SL}_3(\mathbb{Z})$, as well as any other linear group, has uniform exponential growth (see [5]).

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